METALOGIC

An Introduction to the Metatheory of Standard First Order Logic

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$$(A \vee B) \equiv (\sim A \supset B).$$

So we can rewrite $A_1 \vee A_2$ as $\sim A_1 \supset A_2$. Then we have:

If $\models_P A \supset B$, then $\models_P A \supset (\sim A_1 \supset A_2)$ and $\models_P (\sim A_1 \supset A_2) \supset B$.

So we have a formula C_1, viz. $(\sim A_1 \supset A_2)$ such that $\models_P A \supset C_1$, and $\models_P C_1 \supset B$, and C_1 has only $(n-1)$ propositional symbols that occur in A but not in B.

Now repeat the argument on $\models_P C_1 \supset B$ (instead of on $\models_P A \supset B$), and we should have $\models_P C_1 \supset C_2$, and $\models_P C_2 \supset B$, and C_2 has only $(n-2)$ propositional symbols that occur in A but not in B.

Repeat the argument for $(n-2)$ more times, collecting together all the results; we should now have: $\models_P A \supset C_1$, $\models_P C_1 \supset C_2$, $\models_P C_2 \supset C_3$, ... , $\models_P C_{n-1} \supset C_n$, $\models_P C_n \supset B$.

From these, it could be established, via the truth-table for '$\supset$', that $\models_P A \supset C_n$ and $\models_P C_n \supset B$, where C_n has no propositional symbols that occur in A but not in B, and C_n is the required C. (Thanks to Edwin Hung.)

A rigorous proof of this theorem, by mathematical induction on the number of propositional symbols in A but not in B, is given as the answer to the exercise on §27.

21 P's powers of expression. Adequate sets of connectives

We shall prove (Theorem 21.1) that the language P is capable of expressing any truth function, in the following sense:

> To every truth function there corresponds in a natural way a complete truth table. To every complete truth table there corresponds (not necessarily uniquely) a formula of P ('corresponds' in the sense of having that truth table as its truth table).

Examples:

1. To the truth function *material implication* there corresponds the table

$\langle T, T\rangle = T$
$\langle F, T\rangle = T$
$\langle T, F\rangle = F$
$\langle F, F\rangle = T$

To that table there corresponds the formula $p' \supset p''$, among others:

p'	p''	$p' \supset p''$
T	T	T
F	T	T
T	F	F
F	F	T

2. To the nameless truth function to which there corresponds the table

$\langle T, T, T\rangle = F$
$\langle F, T, T\rangle = F$
$\langle T, F, T\rangle = F$
$\langle F, F, T\rangle = F$
$\langle T, T, F\rangle = F$
$\langle F, T, F\rangle = T$
$\langle T, F, F\rangle = F$
$\langle F, F, F\rangle = F$

there corresponds (among others) the formula

$$\sim(\sim(\sim p' \supset \sim p'') \supset \sim\sim p''')$$

thus:

p'	p''	p'''	$\sim(\sim(\sim p' \supset \sim p'') \supset \sim\sim p''')$
T	T	T	F
F	T	T	F
T	F	T	F
F	F	T	F
T	T	F	F
F	T	F	T
T	F	F	F
F	F	F	F

Instead of saying that P is capable of expressing any truth function, we shall say that the set of connectives $\{\sim, \supset\}$ is adequate for the expression of any truth function, since the only connectives in P are $\sim$ and $\supset$. Some other sets of connectives are also adequate; i.e. some languages, that differ from P only in having different connectives, are also capable of expressing any truth function. ('Different connectives' here means connectives that differ in their truth-table definition, not merely in the physical shape of their tokens.)

Theorem 21.1 (which we still have to prove) is our first important metatheorem:

21.1 *The set* $\{\sim, \supset\}$ *is adequate for the expression of any truth function* [*so P can express any truth function*]

The proof of Metatheorem 21.1 is in two stages. We prove first that the set $\{\sim, \wedge, \vee\}$ is adequate (Metatheorem 21.2); then that if the set $\{\sim, \wedge, \vee\}$ is adequate, then the set $\{\sim, \supset\}$ is.

21.2 *The set* $\{\sim, \wedge, \vee\}$ *is adequate*

Proof. Intuitively, the proof consists in showing that, given any complete truth table, we can construct a formula[1] in disjunctive normal form that has that table as its truth table.

A formula is in *disjunctive normal form* (DNF) iff it is a disjunction of conjunctions of single propositional symbols or their negations; counting as degenerate cases of disjunctions / conjunctions single propositional symbols and their negations, and allowing disjunctions / conjunctions of more than two disjuncts / conjuncts and of just one disjunct / conjunct.

Examples (throughout we only include such brackets as are necessary to avoid ambiguity):

1. $(p' \wedge \sim p'' \wedge p''') \vee (p'' \wedge \sim p') \vee (\sim p'''' \wedge p''')$
2. $(p' \wedge \sim p'' \wedge p''') \vee (p'' \wedge \sim p') \vee \sim p''''$
   [$\sim p''''$ counts as a degenerate conjunction with only one conjunct.]
3. $p' \vee p''$
   [Each of p' and p'' counts as a degenerate conjunction.]
4. $p' \wedge p''$
   [This counts as a degenerate disjunction with only one disjunct, viz. the whole formula.]
5. $\sim p'$
   [This counts as a degenerate disjunction of a degenerate conjunction.]

Notice that a formula is in DNF only if

(1) the only connectives that occur in it are connectives for negation, conjunction and disjunction (not necessarily all of these), and

[1] From here to the end of the section 'formula' is used to cover not only formulas of P but also formulas of languages with truth-functional propositional connectives additional to those in P.

(2) negation is over single propositional symbols only, not over any more complicated expressions (e.g. not over conjunctions or disjunctions).

Now the proof can go quite simply:

Let *f* be any arbitrary truth function of an arbitrary number, *n*, of arguments. Write out the complete truth table corresponding to *f*. It will have $n + 1$ columns and 2^n rows. Look at the T's and F's in the last column (i.e. the column that gives the values of the function for the sets of arguments in the corresponding rows). There are three possibilities:

1. The last column is all F's.
2. There is exactly one T in the last column.
3. There is more than one T in the last column.

We show in each case how to construct a formula in DNF having *n* distinct propositional symbols and the same truth table as *f*.

Case 1 (The last column is all F's)

Then

$$p' \wedge \sim p' \wedge p'' \wedge p''' \wedge \ldots \wedge p^n$$

[where p^n is an abbreviation for *p* followed by *n* dashes] is a formula in DNF that has the same truth table as *f*. For $p' \wedge \sim p'$ always gets F, and so therefore does anything of which it is a conjunct.

Case 2 (The last column has just one T)

Go along the row that has the T in its final column. If the first entry in the row is T, write p'; if the first entry is F, write $\sim p'$. If the second entry is T, write p''; if it is F, write $\sim p''$. And so on, as far as and including the *n*th entry. Form the *conjunction* of what you have written (i.e. insert $n - 1$ conjunction signs in the appropriate places). The resulting formula will be in DNF and have the same truth table as the function *f*.

Example: Let *f* be the function of three arguments that has the table

T	T	T	=F
F	T	T	=F
T	F	T	=T
F	F	T	=F

T	T	F	=F
F	T	F	=F
T	F	F	=F
F	F	F	=F

Then the formula

$$p' \wedge {\sim}p'' \wedge p'''$$

is in DNF and has the same truth table as f. It has the value T iff p' has T, p'' has F, and p''' has T; in all other cases it has the value F.

Case 3 (More than one T)

For each row that ends in a T construct a formula as in Case 2. Form the *disjunction* of all these formulas. The resulting formula will be in DNF and have the same truth table as f.

Example: Let f be the function of three arguments that has the table

T	T	T	=F
F	T	T	=F
T	F	T	=T
F	F	T	=F
T	T	F	=F
F	T	F	=T
T	F	F	=F
F	F	F	=T

Then the formula

$$(p' \wedge {\sim}p'' \wedge p''') \vee ({\sim}p' \wedge p'' \wedge {\sim}p''') \vee ({\sim}p' \wedge {\sim}p'' \wedge {\sim}p''')$$

is in DNF and has the same truth table as f. It has the value T in each of the three cases

(1) p' T, p'' F, p''' T
(2) p' F, p'' T, p''' F
(3) p', p'', p''' all F

and the value F otherwise.

This completes the proof of Metatheorem 21.2.

Proof of Metatheorem 21.1 (*The set* $\{\sim, \supset\}$ *is adequate*)

1. Any *conjunction* of two formulas A and B has the same truth table as a formula in which A and B are related by $\sim$ and $\supset$ instead of $\wedge$, thus:

$$(\mathrm{A} \wedge \mathrm{B}) \equiv {\sim}(\mathrm{A} \supset {\sim}\mathrm{B}).$$

Let C be any formula in which $\wedge$ occurs. Then by replacing every subformula of C that has the form $(\mathrm{A} \wedge \mathrm{B})$ by a subformula of the form ${\sim}(\mathrm{A} \supset {\sim}\mathrm{B})$ we get a formula in which $\wedge$ does not occur and that has the same truth table as C.

2. Similarly for $\vee$. Any *disjunction* of two formulas A and B has the same truth table as a formula in which A and B are related by $\sim$ and $\supset$ instead of $\vee$, thus:

$$(\mathrm{A} \vee \mathrm{B}) \equiv ({\sim}\mathrm{A} \supset \mathrm{B}) \quad [\text{i.e. } ([{\sim}\mathrm{A}] \supset \mathrm{B})].$$

3. Let W be any formula in which either $\wedge$ or $\vee$ occurs, or both $\wedge$ and $\vee$ occur. By carrying out successively the replacement operations described in (1) and (2) above, we get a formula W′ in which no connectives other than $\sim$ and $\supset$ occur and that has the same truth table as W.

4. So since the set $\{\sim, \wedge, \vee\}$ is adequate for the expression of any truth function [21.2], so also is the set $\{\sim, \supset\}$.

Q.E.D.

By similar arguments other sets of connectives can also be shown to be adequate.

21.3 *The set* $\{\sim, \vee\}$ *is adequate* [Emil L. Post, 1920]

Proof. Use Metatheorem 21.2 and the tautological schema

$$(\mathrm{A} \wedge \mathrm{B}) \equiv {\sim}({\sim}\mathrm{A} \vee {\sim}\mathrm{B}).$$

21.4 *The set* $\{\sim, \wedge\}$ *is adequate*

Proof. Use Metatheorem 21.2 and the tautological schema

$$(\mathrm{A} \vee \mathrm{B}) \equiv {\sim}({\sim}\mathrm{A} \wedge {\sim}\mathrm{B}).$$

C. S. Peirce in a paper of about 1880 that he did not publish ('A Boolian Algebra with One Constant', *Collected Papers*, iv, §§12–20 [pp. 13–18]) presented a language for Boolean algebra with just one constant 'which serves at the same time as the only sign for compounding terms and which renders special signs for negation, for "what is" and for "nothing" unnecessary'. For our present purpose we can take it as a dyadic connective meaning 'Neither A nor B'. Peirce claimed that it was adequate, but he did not give a rigorous proof of its adequacy. Later, in another unpublished paper, written in 1902 (*Collected Papers*, iv, §265 [p. 216]), he showed that anything that could be expressed by the connective meaning 'Neither A nor B' could equally be expressed using only a connective meaning 'Either not A or not B'. Henry M. Sheffer, without knowing Peirce's result, showed (1912) that all truth functions expressible by means of the primitive connectives ($\sim$, $\vee$) of *Principia Mathematica* could be expressed by either of Peirce's two connectives. Emil L. Post was the first to give a completely general proof of adequacy (for $\{\sim, \vee\}$, in his doctoral dissertation for Columbia University, completed in 1920 and published in the following year: cf. Post, 1920).

21.5 *The set* $\{\downarrow\}$ *is adequate* [C. S. Peirce, *c.* 1880; H. M. Sheffer, 1912. But see comment above.]

A $\downarrow$ B has the value T iff A and B both have the value F. So $p \downarrow q$ can be read as 'Neither p nor q'.

Proof. Use Metatheorem 21.4 and the tautological schemata

$$\sim A \equiv A \downarrow A, \quad (A \wedge B) \equiv (A \downarrow A) \downarrow (B \downarrow B).$$

21.6 *The set* $\{|\}$ *is adequate* [C. S. Peirce, 1902; H. M. Sheffer, 1912. But see comment preceding 21.5.]

The symbol | expresses what is usually called 'the Sheffer stroke function' (for which see the comment preceding 21.5). A|B has the value F iff A and B both have the value T. So $p|q$ can be read as 'Not both p and q' or as 'Either not p, or not q, or not p and not q'.

Proof. Use Metatheorem 21.3 and the tautological schemata

$$\sim A \equiv A|A, \quad (A \vee B) \equiv (A|A)|(B|B).$$

There are other adequate sets, and some inadequate ones.

21.7 *The set* $\{\wedge, \vee\}$ *is inadequate*

The proof (which here we only indicate) is by showing that the negation of a formula cannot be expressed by any combination of propositional symbols, $\wedge$, and $\vee$. Let P′ be a language just like P, except that it has the connectives $\wedge$ and $\vee$ in place of the connectives $\sim$ and $\supset$. It is shown that (1) no formula of P′ that consists of just one [occurrence of a] symbol can have the value T when all its component propositional symbols have the value F. Then it is shown that (2) *if* this is true of *every* formula of P′ with fewer than m [occurrences of] symbols, *then* it is also true of every formula of P with exactly m symbols. It follows that *no* formula of P′ can have the value T when all its component propositional symbols have the value F, and therefore that P′ cannot express negation. [This type of proof is known as proof by (strong) mathematical induction, about which more later.]

21.8 *The set* $\{\wedge, \supset\}$ *is inadequate*

Proof similar to that for 21.7.

21.9 *The set* $\{\supset, \vee\}$ *is inadequate*

Proof similar to that for 21.7.

Not every set that has $\sim$ as a member is adequate, and not every set that does not is inadequate:

21.10 *The set* $\{\sim, \equiv\}$ *is inadequate*

The proof is similar to that for 21.7, but in this case we show that material implication, for example (we could equally well take conjunction, or disjunction), cannot be expressed by any combination of propositional symbols, $\sim$, and $\equiv$. For the truth table for material implication has four rows and a final column with three T's and one F; while any four-rowed truth table for any formula with no connectives other than $\sim$ and $\equiv$ must have either all T's in its final column, or all F's, or two T's and two F's. (This is rigorously proved by mathematical induction in the answer to exercise 2 of §27, p. 90.)

21.11 *Material implication and exclusive disjunction together are adequate*

[There is no agreed symbol for exclusive disjunction. The truth table for it is

A	B	A excl. disj. B
T	T	F
F	T	T
T	F	T
F	F	F

It can be seen that exclusive disjunction has the same truth table as negated material equivalence. So we shall use the symbol $\not\equiv$ for exclusive disjunction.]

Proof. Use 21.1 and the tautological schema

$$\sim A \equiv (A \not\equiv (A \supset A)).$$

21.12 *The only dyadic connectives that are adequate by themselves are* | *and* ↓ [Żyliński, 1924]

Proof. Suppose there was another connective. Let it be ∗. We work out what its truth table would have to be, row by row:

A	B	A ∗ B
T	T	?
F	T	?
T	F	?
F	F	?

If the entry in the first row were T, then any formula built up using only ∗ would take the value T when all its propositional symbols took the value T. So no combination could express the negation of A. So the entry for the first row must be F. Similarly, the entry for the last row must be T.

This gives us:

A	B	A ∗ B
T	T	F
F	T	?
T	F	?
F	F	T

If the second and third entries were both Ts, ∗ would have the same table as | (and so would be the same connective as |, in all but the physical shape of its tokens, which is not important from a theoretical point of view).[1] If they were both Fs, ∗ would be the same as ↓. That leaves just two possibilities to consider, viz. (1) second entry T, third entry F, and (2) second entry F, third entry T. In the first case we would have

$$A * B \equiv \sim A.$$

In the second case we would have

$$A * B \equiv \sim B.$$

[1] A truth-functional propositional connective is a *meaningful* symbol, not a merely formal symbol.

In either case ∗ would be definable in terms of ~. But ~ is not adequate by itself, because the only functions of one argument definable from it are negation and identity. I.e. starting from a formula A and using only negation we can get formulas that are truth-functionally equivalent to A and formulas that are truth-functionally equivalent to ~A, but nothing else:

A
~A
~~A
~~~A
~~~~A
. . .

We cannot define in terms of ~ alone either of the other two truth functions of one argument (§16, pp. 50, 51 above), viz.

$$\begin{cases} f_1(T) = T \\ f_1(F) = T \end{cases}$$

and

$$\begin{cases} f_4(T) = F \\ f_4(F) = F \end{cases}$$

So any dyadic connective that is not the same as either | or ↓ is inadequate by itself.

Q.E.D.

22 A deductive apparatus for P: the formal system PS. Definitions of *proof in PS, theorem of PS, derivation in PS, syntactic consequence in PS, proof-theoretically consistent set of PS*

Treat this section as though it followed on directly from the end of §18. Pretend that you know nothing of the contents of §§19–21, i.e. nothing about any interpretation of P.

We now specify a deductive apparatus for the formal language P, viz. a set of axioms and a rule of inference. We call the resulting formal system the system PS (propositional system).